Revisiting two classical results on graph spectra

Vladimir Nikiforov
Department of Mathematical Sciences, University of Memphis,
Memphis TN 38152, USA

February 2, 2008

Abstract

Let $\mu(G)$ and $\mu_{\min}(G)$ be the largest and smallest eigenvalues of the adjacency matrix of a graph G. Our main results are:

(i) If H is a proper subgraph of a connected graph G of order n and diameter D, then

$$\mu(G) - \mu(H) > \frac{1}{\mu^{2D}(G)\, n}.$$

(ii) If G is a connected nonbipartite graph of order n and diameter D, then

$$\mu(G) + \mu_{\min}(G) > \frac{2}{\mu^{2D}(G)\, n}.$$

These bounds have the correct order of magnitude for large μ and D.

Keywords: *smallest eigenvalue, largest eigenvalue, diameter, connected graph, bipartite graph*

1 Introduction

Our notation is standard (e.g., see [2], [3], and [5]). In particular, unless specified otherwise, all graphs are defined on the vertex set $[n] = \{1, ..., n\}$ and $\mu(G)$ and $\mu_{\min}(G)$ stand for the largest and smallest eigenvalues of the adjacency matrix of a graph G.

The aim of this note is to refine quantitatively two well-known results on graph spectra. The first one, following from Frobenius's theorem on nonnegative matrices, asserts that if H is a proper subgraph of a connected graph G, then $\mu(G) > \mu(H)$. The second one, due to H. Sachs [7], asserts that if G is a connected nonbipartite graph, then $\mu(G) > -\mu_{\min}(G)$.

Our main result is the following theorem.

Theorem 1 *If H is a proper subgraph of a connected graph G of order n and diameter D, then*

$$\mu(G) - \mu(H) > \frac{1}{\mu^{2D}(G)\, n}. \tag{1}$$

It can be shown that, for large μ and D, the right-hand of (1) gives the correct order of magnitude; examples can be constructed as in the proofs of Theorems 2 and 3 below.

Theorem 2 *If G is a connected nonbipartite graph of order n and diameter D, then*

$$\mu(G) + \mu_{\min}(G) > \frac{2}{\mu^{2D}(G)\, n}. \tag{2}$$

Moreover, for all $k \geq 3$, $D \geq 4$, and $n = D + 2k - 1$, there exists a connected nonbipartite graph G of order n and diameter D with $\mu(G) > k$, and

$$\mu(G) + \mu_{\min}(G) < \frac{4}{(k-1)^{2D-4}}.$$

Theorem 2 shows that $\mu(G) + \mu_{\min}(G)$ can be extremely small, although G is nonbipartite and connected. Here is another viewpoint to this fact.

Theorem 3 *Let $0 < \varepsilon < 1/16$. For all sufficiently large n, there exists a connected graph G of order n with $\mu(G) + \mu_{\min}(G) < n^{-\varepsilon n}$ such that, to make G bipartite, at least $(1/16 - \varepsilon)\, n^2$ edges must be removed.*

The picture is completely different for regular graphs. In [4] it is proved that if G is a connected nonregular graph of order n, size m, diameter D, and maximum degree Δ, then

$$\Delta - \mu(G) > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}.$$

This result and Theorem 1 help deduce the following theorems; we omit their straightforward proofs.

Theorem 4 *If H is a proper subgraph of a connected regular graph G of order n and diameter D, then*

$$\mu(G) - \mu(H) > \frac{1}{n(D+1)}.$$

Theorem 5 *If G is a connected regular nonbipartite graph of order n and diameter D, then*

$$\mu(G) + \mu_{\min}(G) > \frac{2}{n(2D+1)}.$$

Theorem 6 *If G is a connected, nonregular, nonbipartite graph of order n, diameter D, and maximum degree Δ, then*

$$\Delta + \mu_{\min}(G) > \frac{1}{n(D+1)} + \frac{1}{\mu^{2D}(G)\, n}.$$

Note that the last two theorems give a fine tuning of a result of Alon and Sudakov [1].

2 Proofs

Our proof of Theorem 1 stems from a result of Schneider [8] on eigenvectors of irreducible nonnegative matrices; for graphs it reads as: if G is a connected graph of order n and $x_{\min}$, $x_{\max}$ are minimal and maximal entries of an eigenvector to $\mu(G)$, then

$$\frac{x_{\min}}{x_{\max}} \geq \mu^{-n+1}(G).$$

We reprove this inequality in a more flexible form that sheds some extra light on the original matrix result of Schneider as well. Hereafter we write $dist(u,v)$ for the length of a shortest path joining the vertices u and v.

Proposition 7 *If G is a connected graph of order n and $(x_1,\ldots,x_n)$ is an eigenvector to $\mu(G)$, then*

$$\frac{x_i}{x_j} \geq (\mu(G))^{-dist(i,j)} \tag{3}$$

for every two vertices $i,j \in V(G)$.

Proof Clearly we can assume that $i \neq j$. For convenience we also assume that $i=1$ and the vertices $(1,\ldots,j)$ form a path joining 1 to j. Then, for all $u=1,\ldots,j-1$, we have

$$\mu x_u = \sum_{uv\in E(G)} x_v \geq x_{u+1};$$

hence, (3) follows by multiplying all these inequalities. □

We shall need also the following simple bound.

Proposition 8 *If G is a connected graph of order $n \geq 3$ and diameter D, then $\mu^D(G) > n/\sqrt{3}$.*

Proof Note that every two vertices can be joined by a walk of D or $D+1$ vertices. Hence, letting $w_k(G)$ be the number of walks of k vertices, we find that $w_D(G)+w_{D+1}(G) \geq n^2$; therefore, by a result in [6], $\mu^{D-1}(G)+\mu^D(G) \geq n$. Since $\mu(G) > \sqrt{2}$, we see that

$$\sqrt{3}\mu^D(G) > \frac{1}{\sqrt{2}}\mu^D(G)+\mu^D(G) \geq \mu^{D-1}(G)+\mu^D(G) \geq n,$$

completing the proof. □

Proof of Theorem 1 Since $\mu(H) \leq \mu(H')$ whenever $H \subset H'$, we may assume that H is a maximal proper subgraph of G, that is to say, $V(H)=V(G)$ and H differs from G in a single edge uv. Our proof is split into two cases: (a) H connected; (b) H disconnected.

Case (a): H is connected.

In this case we shall prove a stronger result than required, namely

$$\mu(G) - \mu(H) > \frac{2}{\mu^{2D}(G)\,n}. \tag{4}$$

Our first goal is to prove that, for every $w \in V(H)$,

$$dist_H(w,u) + dist_H(w,v) \leq 2D. \tag{5}$$

Let $w \in V(H)$ and select in H shortest paths $P(u,w)$ and $P(v,w)$ joining u and v to w. Let $Q(u,x)$ and $Q(v,x)$ be the longest subpaths of $P(u,w)$ and $P(v,w)$ having no internal vertices in common. If $s \in Q(u,x)$ or $s \in Q(v,x)$, we obviously have

$$dist_H(w,s) = dist_H(w,x) + dist_H(s,x). \tag{6}$$

The paths $Q(u,x)$, $Q(v,x)$ and the edge uv form a cycle in G; write k for its length. Assume that $dist(v,x) \geq dist(u,x)$ and select $y \in Q(v,x)$ with $dist_H(x,y) = \lfloor k/2 \rfloor$. Let $R(w,y)$ be a shortest path in G joining w to y; clearly the length of $R(w,y)$ is at most D. If $R(w,y)$ does not contain the edge uv, it is a path in H and, using (6), we find that

$$\begin{aligned} D &\geq dist_G(w,y) = dist_H(w,y) = dist_H(w,x) + \lfloor k/2 \rfloor \\ &= dist_H(w,x) + \left\lfloor \frac{dist_H(x,u) + dist_H(x,v) + 1}{2} \right\rfloor \\ &\geq dist_H(w,x) + \frac{dist_H(x,u) + dist_H(x,v)}{2} = \frac{dist_H(w,u) + dist_H(w,v)}{2}, \end{aligned}$$

implying (5). Let now $R(w,y)$ contain the edge uv. Assume first that v occurs before u when traversing $R(w,y)$ from w to y. Then

$$\begin{aligned} dist_H(w,u) + dist_H(w,v) &\leq 2dist_H(w,x) + dist_H(x,u) + dist_H(x,v) \\ &\leq 2\left(dist_H(w,x) + dist_H(x,v)\right) < dist_G(w,y) \leq 2D, \end{aligned}$$

implying (5). Finally, if u occurs before v when traversing $R(w,y)$ from w to y, then

$$\begin{aligned} D &\geq dist_G(w,y) \geq dist_H(w,u) + 1 + dist_H(v,y) \\ &= dist_H(w,x) + dist_H(x,u) + 1 + dist_H(v,y) = dist_H(w,x) + \lceil k/2 \rceil \\ &\geq dist_H(w,x) + \frac{dist_H(x,u) + dist_H(x,v)}{2} = \frac{dist_H(w,u) + dist_H(w,v)}{2}, \end{aligned}$$

implying (5). Thus, inequality (5) is proved in full.

Let now $\mathbf{x} = (x_1, ..., x_n)$ be a unit eigenvector to $\mu(H)$ and let x_w be a maximal entry of $\mathbf{x}$. In view of (3) and (5), we have

$$\frac{x_u x_v}{x_w^2} \geq \frac{1}{\mu^{dist(u,w)+dist(v,w)}(H)} \geq \frac{1}{\mu^{2D}(H)}.$$

Hence, in view of $x_w^2 \geq 1/n$, we see that

$$\mu(G) \geq 2 \sum_{ij \in E(G)} x_i x_j = 2x_u x_v + \mu(H) \geq \frac{2x_w^2}{\mu^{2D}(H)} + \mu(H) > \frac{2}{\mu^{2D}(G)\, n} + \mu(H),$$

completing the proof of (4) and thus of (1).

Case (b): H is disconnected.

Since G is connected, H is union of two connected graphs H_1 and H_2 such that $v \in H_1$, $u \in H_2$. Assume $\mu(H) = \mu(H_1)$, set $|H_1| = k$, and let $\mathbf{x} = (x_1, ..., x_k)$ be a unit eigenvector to $\mu(H_1)$. Since any maximal entry of $\mathbf{x}$ is at least $k^{-1/2}$ and *diam* $H_1 \leq$ *diam* $G \leq D$, Proposition 7 implies that $x_v \geq \mu^{-D}(H)\, k^{-1/2}$. Set $t = \mu^{-D}(H)\, k^{-1/2}$ and consider the unit vector

$$(y_1, ..., y_k, y_u) = \left(x_1\sqrt{1-t^2}, ..., x_k\sqrt{1-t^2}, t\right).$$

Then

$$\begin{aligned}
\mu(G) \geq \mu(H_1 + u) \geq 2 \sum_{ij \in E(H_1+u)} y_i y_j &\geq 2t \sum_{uj \in E(H_1+u)} y_j + 2\left(1-t^2\right) \sum_{ij \in E(H_1)} x_i x_j \\
&\geq 2t\sqrt{1-t^2}\, x_v + \left(1-t^2\right)\mu(H) = \frac{1}{\mu^{2D}(H)\, k}\left(2\sqrt{1-\frac{1}{\mu^{2D}(H)\, k}} - 1\right) + \mu(H).
\end{aligned}$$

For $k \geq 3$, Proposition 8 implies that

$$\begin{aligned}
\frac{1}{\mu^{2D}(H)\, k}\left(2\sqrt{1-\frac{1}{\mu^{2D}(H)\, k}} - 1\right) &> \frac{1}{\mu^{2D}(H)\, k}\left(2\sqrt{1-\frac{3}{k^3}} - 1\right) \\
&> \frac{1}{\mu^{2D}(H)(k+1)} > \frac{1}{\mu^{2D}(G)\, n}.
\end{aligned}$$

Finally, if $k = 3$, then $\mu(H_1) = 1$, $\mu(G) \geq \sqrt{2}$, $D \geq 2$, and $n \geq 3$; hence,

$$\mu(G) - \mu(H) \geq \sqrt{2} - 1 > \frac{1}{3\left(\sqrt{2}\right)^4} \geq \frac{1}{\mu^{2D}(G)\, n},$$

completing the proof. □

Proof of Theorem 2 Let $\mathbf{x} = (x_1, ..., x_n)$ be an eigenvector to $\mu_{\min}(G)$ and let $V_1 = \{u : x_u < 0\}$. Let H be the maximal bipartite subgraph of G, containing all edges with exactly one vertex in V_1. It is not hard to see that H is connected proper subgraph of G, $V(H) = V(G)$, and $\mu_{\min}(H) < \mu_{\min}(G)$. Finally, let H' be a maximal proper subgraph of G containing H. We have

$$\mu(G) + \mu_{\min}(G) \geq \mu(G) + \mu_{\min}(H) = \mu(G) - \mu(H) \geq \mu(G) - \mu(H').$$

and (2) follows from case (a) of the proof of Theorem 1.

To construct the required example, set $G_1 = K_3$, $G_2 = K_{k,k}$, join G_1 to G_2 by a path P of length $n-2k-2$, and write G for the resulting graph; obviously G is of order n and diameter $n-2k+1$. Set $\mu = \mu(G)$ and note that $\mu(G) > k$. Let $V(G_1) = \{u_1, u_2, v_1\}$ and $P = (v_1, \ldots, v_{n-2k-1})$, where $v_{n-2k-1} \in V(G_2)$. Let $\mathbf{x}$ be a unit eigenvector to $\mu(G)$ and assume that the entries $x_1, x_2, x_3, \ldots, x_{n-2k+1}$ correspond to $u_1, u_2, v_1, \ldots, v_{n-2k-1}$. Clearly $x_1 = x_2$, and so, from $\mu x_2 = x_2 + x_3$, we find that $x_1 = x_2 = x_3/(\mu - 1)$. Furthermore,

$$\mu x_3 = 2x_2 + x_4 = \frac{2x_3}{\mu - 1} + x_4 < x_3 + x_4,$$

and by induction we obtain $x_i < (\mu - 1) x_{i+1}$ for all $3 \leq i \leq n - 2k$. Therefore,

$$x_1 = x_2 \leq (\mu - 1)^{-n+2k+1} x_{n-2k+1} < (k-1)^{-D+2},$$

and by Rayleigh's principle we deduce that

$$\mu(G) + \mu_{\min}(G) \leq 4x_1x_2 < \frac{4}{(k-1)^{2D-4}},$$

completing the proof. □

Proof of Theorem 3 Set $r = \lceil n/4 \rceil + 1$, $s = \lceil (1/2 - \varepsilon) n \rceil$, select $G_1 = K_{r,r}$, $G_2 = K_s$, join G_1 to G_2 by a path P of length $n - 2r - s + 1$ and write G for the resulting graph. Note first that, to make G bipartite, we must remove at least

$$\binom{s}{2} - \left\lfloor \frac{s^2}{4} \right\rfloor \geq \frac{s^2}{4} - \frac{s}{2} > \frac{(1/2 - \varepsilon)^2 n^2}{4} - \frac{s}{2} \geq \left(\frac{1}{16} - \varepsilon\right) n^2$$

edges, for n large enough. Note also that

$$n - 2\left\lceil \frac{n}{4} \right\rceil - 2 - \left\lceil \left(\frac{1}{2} - \varepsilon\right) n \right\rceil + 1 > n - \frac{n}{2} - \left(\frac{1}{2} - \varepsilon\right) n - 4 = \varepsilon n - 4.$$

so the length of P is greater than $\varepsilon n - 4$.

Let $\mathbf{x}$ be a unit eigenvector to $\mu(G)$. Clearly the entries of $\mathbf{x}$ corresponding to vertices from $V(G_1) \backslash V(P)$ have the same value α. Like in the proof of Theorem 2, we see that $\alpha < (n/4)^{-\varepsilon n + 5}$. Hence, by Rayleigh's principle, for n large enough, we deduce that

$$\mu(G) + \mu_{\min}(G) \leq 4\alpha^2 \binom{s}{2} < (n/4)^{-2\varepsilon n + 10} \frac{n^2}{2} < (n/4)^{-2\varepsilon n + 12} < n^{-\varepsilon n},$$

completing the proof. □

Acknowledgment The author is indebted to Béla Bollobás for his kind support and to Sebi Cioabă for interesting discussions.

References

[1] N. Alon, B. Sudakov, Bipartite subgraphs and the smallest eigenvalue, *Combin. Probab. Comput.* 9 (2000) 1-12.

[2] B. Bollobás, *Modern Graph Theory,* Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[3] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs,* VEB Deutscher Verlag der Wissenschaften, Berlin, 1980, 368 pp.

[4] S. Cioabă, D. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, to appear in *J. Combin. Theory Ser B.*

[5] R. Horn, C. Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, 1985, xiii+561 pp.

[6] V. Nikiforov, Walks and the spectral radius of graphs, *Linear Algebra Appl.* 418 (2006), 257-268

[7] H. Sachs, Beziehungen zwischen den in einem Graphen enthalteten Kreisen und seinem charakteristischen Polynom, *Publ. Math. Debrecen* 11 (1964) 119–134.

[8] H. Schneider, Note on the fundamental theorem on irreducible non-negative matrices, *Proc. Edinburgh Math. Soc.* 11 (1958/1959) 127–130.